Response to "Comment on 'Penetrability of a one-dimensional Coulomb potential' " by Roger G Newton

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COMMENT

Response to "Comment on 'Penetrability of a one-dimensional Coulomb potential'" by Roger G Newton

Marcos Moshinsky
Instituto de Física, UNAM, Apdo. Postal 20-364, 01000 México DF, Mexico

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Abstract. By expressing in matrix form the one-dimensional Hamiltonians where the potentials are proportional to $|x|^{-1}$ and x^{-1}, I show that we are dealing with two very different problems. Thus, in my viewpoint, the criticism of R G Newton of the paper mentioned in the title, which is based on results relating to $|x|^{-1}$, does not apply to a potential of the form x^{-1}.

I was flattered that a distinguished expert in scattering theory, Professor Roger G Newton, who is in addition the current Editor of the Journal of Mathematical Physics, paid attention to my paper on 'Penetrability of the one-dimensional Coulomb potential' [1].

For brevity, I will concentrate on his criticism of my analysis for positive energies only. As he remarks I have mentioned all the literature (eight papers) that I could find in the American Journal of Physics on the $|x|^{-1}$ problem, but these references did not seem relevant to my viewpoint. To make this clear, I will first consider the matrix representation of the problem, which, as we know since Schrödinger, is equivalent to the operator form in configuration space.

For a complete set of states I take the kets of the one-dimensional oscillator in the full interval $-\infty \leqslant x \leqslant \infty$, i.e.

$$|n\rangle = A_n H_n(x) \exp(-x^2/2) \tag{1}$$

where I use units in which $\hbar$, the particle mass m and the frequency ω of the oscillator are taken to be unities, with $H_n(x)$ being a Hermite polynomial and A_n the appropriate normalization constant.

I designate my Hamiltonian as

$$\mathcal{H} = \frac{p^2}{2} + V(x) \tag{2}$$

where later on I take

$$V(x) = \frac{\alpha}{|x|} \qquad \text{or} \qquad V(x) = \frac{\alpha}{x} \tag{3}$$

with α being $-(me^4/\hbar^3\omega)^{1/2}$.

For a matrix formulation, I need to consider the elements

$$\langle m|\mathcal{H}|n\rangle = -\tfrac{1}{2}[(n+2)(n+1)]^{1/2}\delta_{m,n+2} + \tfrac{1}{4}[2n+1]\delta_{mn} - \tfrac{1}{2}[n(n-1)]^{1/2}\delta_{m,n-2}$$
$$+ A_m A_n \int_{-\infty}^{\infty} H_m(x) V(x) H_n(x) \exp(-x^2) \mathrm{d}x \tag{4}$$

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What happens when $V(x) = \alpha/|x|$? We immediately see that the last integral in (4) diverges when both m and n are even, because the integrand becomes infinite at the origin $x = 0$.

The only way to get out of this dilemma is to castrate our Hilbert space so that m, n are always odd, and thus vanish at the origin. This is precisely the point of view advocated by Professor Newton, as well as in all the references on the subject that were mentioned for $|x|^{-1}$ in my paper and in his response.

On the other hand, if $V(x) = \alpha/x$, no problem appears because when both m and n are even (and also when both are odd), the integrand in (4) is an odd function of x and the principal value of its integral in the interval $-\infty \leqslant x \leqslant \infty$ vanishes. Thus we can take the full Hilbert space $m, n = 0, 1, 2, 3, \ldots$, and the only terms that contribute to the matrix elements of the potential are those in which m is even and n is odd or vice versa, and in that case one of the Hermite polynomials has only odd powers of x, and so we can take x out as a factor to cancel the potential x^{-1} and get a very definite integral.

The reader may argue that the fact that a matrix representation of the Hamiltonian (2) with $V(x) = (\alpha/x)$ is well defined, tell us nothing about the penetrability of the barrier. This argument is not valid because Filippov [2] and Smirnov *et al* [3], among others, have shown us how to use the representation of the Hamiltonian in a harmonic oscillator basis to solve scattering problems. They, of course, carried out their analysis in the radial variable r in the interval $0 \leqslant r \leqslant \infty$, but their reasoning is such that it can be easily extended to one-dimensional problems. In fact, as Professor Smirnov is now at my Institution, we plan to carry out this analysis in the future but it will require extensive computing and thus cannot be given here.

There is another problem with Professor Newton's comment, which concerns the way he stresses that the *irregular* solution is not acceptable for an equation which includes the point at which this solution or its derivative diverge. To make my point of view clear on this matter I need to relate a bit of history.

In 1951, when I was fresh out of Graduate School in Princeton, where I worked with Professor Wigner, I was trying to develop a simple formulation of his R-matrix theory. I came to the conclusion that this could be done by proposing for a nuclear reaction, in the center-of-mass frame, a wavefunction of two components,

$$\Psi = \begin{bmatrix} \psi_1(r, t) \\ \psi_2(t) \end{bmatrix} \tag{5}$$

where the first component represented the relative motion of the two particles and the second a compound state. The interaction was introduced through a boundary condition at the point of contact of the two particles, i.e. $r = 0$, through considerations of conservation of probability [4]. Due probably to my youth at the time, I had no compunction of using both *regular* and *irregular* solutions for the s-wave, even if I had to consider these solutions and their derivatives at $r = 0$.

Of course this did not involve a mathematical problem as, when one gets rid of the factor $(1/r)$ in the solution, which can be done with r^2 in the volume element, one is left with perfectly bounded solutions at $r = 0$ of the form

$$\sin kr, \quad \cos kr. \tag{6}$$

Nevertheless, there was the *unthinkable* that I had used the irregular solution up to $r = 0$, and yet obtained perfectly reasonable results for the R-matrix and, in fact, dynamical equations that allowed me to study the time-dependent behaviour of the solutions [4], that are of interest even today.

The reader may argue that the previous considerations do not deal with the Coulomb problem, but I wish to turn now to the latter.

Only a couple of years after the paper appeared in *Phys. Rev.* [4], I turned to the problem not of a neutron impinging on a nucleus but of a proton or another nucleus, and wanted to apply the same type of formalism. This appeared in 1953 in *Revista Mexicana de Física*, in Spanish [5].

Even then I became aware of the fact that, contrary to the neutral case [4], I had now irregular solutions bounded at the origin but whose derivative had a logarithmic ∞. Fortunately I had read what, at the time, was a recent article of Bethe [6] on 'Theory of effective range in nuclear scattering', where he found the same difficulty when discussing proton–proton scattering (see, for example, equation (50) of the paper mentioned), and solved it by taking the difference betweeen the derivatives of two wave functions with different energies.

Using this type of reasoning as a basis, I considered the type of difference given in equation (4.3) of the paper under discussion [1] and established the boundary condition that led to an R-matrix for two charged particles colliding at the point of coincidence and forming a compound system [5].

The reader can not be expected to find a number of the Revista Mexicana de Física of 1953 and read in it an article in Spanish. Fortunately though, because of our interest in heavy-ion reactions, the analysis is repeated in a paper entitled 'Dynamical model for heavy ion reactions with a single resonance' [7], and in equations (6a), (13b), (14) of this paper one sees how the paradox of the logarithmic ∞ in the derivative of the wavefunction can be resolved; it is the same procedure that was used in equation (4.3) of the paper under discussion [1].

It was when I found a physical problem, which was mentioned in the introduction to the article under discussion [1], which required an answer on whether the potential x^{-1} was or was not penetrable, that I got interested in the one-dimensional Coulomb problem. I remembered my previous publication in which I had used both regular and irregular solutions, and proceeded to write the paper that caused the comment of Professor Newton.

I will concede that there are problems, even in non-relativistic quantum mechanics, where difference of opinion still prevails. I hope that the interested reader will analyse not only my original article [1] on 'Penetrability of the one-dimensional Coulomb potential' but also the criticism of Professor Newton and my response. I would only like to end with a note familiar to film makers: Nothing adds more to the revenue of a motion picture than forbidding people to see it because it is obscene.

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